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Review of Mathematical Foundations of Quadratic Optimization Problem

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ABSTRACT

This paper presents a condensed study of the mathematical foundations of quadratic optimization, a class of problems positioned between linear and general nonlinear programming. Quadratic functions are analyzed through linear algebra, matrix theory, and convexity to characterize optimal solutions in both unconstrained and constrained settings, with particular attention to the Karush–Kuhn–Tucker (KKT) conditions and the Mangasarian–Fromovitz constraint qualification. Solution strategies including the quadratic penalty method and the augmented Lagrangian approach are examined, together with the quadratic fractional optimization problem (QFOP) and its multi-objective extension (MOQOP), solved via a fuzzy goal programming approach. A comparative discussion of the three methodologies is provided, along with their applications in finance, engineering, machine learning, and transportation.

Keywords Quadratic Optimization, Convexity, KKT Conditions, Mangasarian–Fromovitz Qualification, Penalty Function, Fractional Optimization, Multi-Objective Optimization

1. INTRODUCTION

Optimization identifies, from a set of feasible alternatives, the point at which a real- valued objective function attains its minimum or maximum value. Linear optimization restricts both the objective and the constraints to linear forms; quadratic optimization (QOP) extends this by permitting a quadratic term in the objective while usually retaining linear constraints, thereby combining richer modelling flexibility with continued analytical tractability. A general optimization problem may be stated as

$$\text{Optimize } f(x) = f(x_1, x_2, x_3, \ldots x_n) \text{ s.t. } g_i(x) \leq, \geq, = b_i,\ x_j \geq 0,\ i = 1,\ldots,m,\ j = 1,\ldots,n \qquad (1)$$

An n-variable quadratic function is written in matrix form as

$$f(x) = \tfrac{1}{2} x^T Q x + c^T x + r \qquad (2)$$

where $Q \in R^{n\times n}$ is the symmetric Hessian matrix, $c \in R^n$ the linear-coefficient vector and $r \in R$ a constant. Because its second derivatives are constant, thus

$$\nabla f(x) = Qx + c,\ \nabla^2 f(x) = Q \qquad (3)$$

1.1 Linear Algebra, Convexity and Optimality

The eigenvalues of Q govern the curvature of f .If Q is positive definite implies f is strictly convex with a unique global minimizer , If Q positive is semi-definite implies f is convex, possibly with multiple minimizers and if Q is indefinite then

f is non-convex, admitting saddle points and multiple local optima. Formally, $f: R^n \rightarrow R$ is convex when

$$F(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y), \text{ for all } x, y \in R^n,\ \lambda\in[0,1] \qquad (4)$$

For twice-differentiable functions this is equivalent to positive semi-definiteness of the Hessian, which for a quadratic function reduces to positive semi-definiteness of Q. Convex QOPs enjoy three properties of central importance which are illustrated below

Every local minimum is global.

First-order conditions are both necessary and sufficient.

Strong duality holds under mild regularity conditions.



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1.2 Unconstrained and Constrained Quadratic Optimization

Minimizing f(x) =½xTQx+cTx over Rn gives the stationary condition

$$\nabla f(x^*) = Q\,x^* + c = 0 \qquad (5)$$

so that, when Q is non-singular, x*=−Q-1c, which is the unique global minimizer whenever Q is positive definite. When the linear constraints Ax=b, Gx ≤h are imposed, the problem

$$\text{Min } ½xTQx+cTx \text{ s. t. } Ax=b,\ G\,x \le h \qquad (6)$$

is characterized through the Karush–Kuhn–Tucker (KKT) conditions. Stationery Qx+c+ATλ+GTμ=0, primal feasibility, dual feasibility μ≥0, and complementary slackness μi (Gx−h) i=0 for every is developed in full generality in Section 3.

2. Literature Review

Quadratic programming has been studied for over six decades; standard treatments appear in the texts of Nocedal and Wright [48], Boyd and Vandenberghe [54], Dostál [53], and Bazaraa, Sherali and Shetty [55].

2.1 Early foundations (1950's–1960's)

Barankin and Dorfman [5,6] and Beale [7,8] gave some of the earliest quadratic programming procedures; Frank and Wolfe [24], Wolfe [37], Hildreth [29], and Graves [28] developed simplex-type and pivoting algorithms, while Boot [14] and Dantzig [17] connected quadratic programming to the Kuhn–Tucker framework and the Wolfe–Markowitz algorithm respectively. Cottle [16], Dikin [18], and Abrams and Ben Israel [1] extended the theory to convexity characterizations, iterative solution schemes, and complex quadratic programming.

2.2 Structural and algorithmic advances (1970's–1980's)

Majthay [32] and Fletcher [21] established optimality conditions and general solution algorithms , Eaves [20] and Cabot and Francis [15] examined (non-)convex quadratic minimisation, while Balas [4] used generalized polar for the non-convex case. Goldfarb [25] extended Newton's method to quadratic problems , Goldfarb and Idnani [26] , Best [11], and Dussault et al. [19] developed numerically stable dual, equivalence, and constrained iterative methods respectively.

2.3 Interior-point and complexity results (1989–1999)

Monteiro and Adler [33] and Ye and Tse [38] extended interior-point and Karmarkar-type methods to convex quadratic programming , Jensen and King [31] proposed a decomposition method, Best and Chakravarti [12] characterised the stability of linearly constrained convex quadratic programs. Pardalos [34] and Pardalos and Vavasis [35] established global-optimization algorithms and NP-hardness results for indefinite quadratic programming, while Sun [36] and Fletcher [22] proved convergence for affine-scaling and degeneracy-resolving methods. Ben-Tal and Teboulle [9] established hidden convexity for indefinite quadratic ally constrained problems, Adcock and Meade [2] incorporated transaction costs into portfolio quadratic optimization, Hoch Baum and Hong [30] gave strongly polynomial algorithms, Boland [13] developed a dual active-set method, and Berkelaar et al. [10] , Goemans and Williamson [39] contributed basis-identification and semidefinite-programming approximation results.

2.4 Recent developments (2000–present)

Arioli [3] and Fletcher [23] refined sparse and indefinite quadratic programming methods using QR and reduced-Hessian updates; Conn, Gould and Toint [42] and Gould and Toint [27, 41] surveyed trust-region methods and compiled a comprehensive quadratic-programming bibliography. Fan and Kalaba [40] modelled manufacturing processes as linear-quadratic optimization problems, Chakra borty and Gupta [43] proposed a fuzzy transformation for multi-objective linear fractional programming — a precursor to the fuzzy goal programming methodology used in Sections 5 and 6 below. Pramanik et al. [44] extended this idea to quadratic bi-level programming, Singh and Sharma [45,46,47] developed least-squares, bounded-solution, and fuzzy-goal-programming approaches to quadratic and quadratic fractional programming problems, directly informing the methodology adopted in this paper.



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3. Necessary Optimality Conditions: KKT and Mangasarian–Fromovitz

3.1 The KKT System

Consider the constrained QOP

$$\text{Min } f(x) = \tfrac{1}{2}x^TQx+c^Tx \text{ s.t. } a_i^Tx \le b_i \ (i=1,\dots,m),\ d_j^Tx = e_j \ (j=1,\dots,p) \quad (7)$$

with feasible set $\Omega=\{x\in R^n : Ax \le b,\ Dx=e\}$. The associated Lagrangian is

$$L(x, y, \lambda, \mu)=\tfrac{1}{2}x^TQx+c^Tx+\lambda^T(Ax+y-b)+\mu^T(0-x) \quad (8)$$

and the first-order KKT conditions read

$$-Qx-A^T\lambda+\mu = c,\ \lambda^T(Ax-b)=0,\ x^T\mu=0,\ Ax-b\le 0,\ x\ge 0,\ \mu\ge 0,\ \lambda\ge 0 \quad (9)$$

A point satisfying (9) is a KKT point. These conditions are necessary for every QOP, convex or not, but sufficient for optimality only in the convex case.

3.2 Structural Properties of Q

Few important points that are used repeatedly in the study are

i. λ is an eigenvalue of Q if and only if $\det(Q-\lambda I)=0$.

ii. $f(x)=\tfrac{1}{2}x^TQx+c^Tx+d$ is convex (concave) iff Q is positive (negative) semi-definite, strictly so iff Q is positive (negative) definite, and neither convex nor concave when Q is indefinite.

iii. $\lambda_{max}=\lambda_1\ge\dots\ge\lambda_n=\lambda_{min}$ for the eigenvalues of a symmetric Q,

$$\alpha\Sigma_i x_i^2+c^Tx+d \le \tfrac{1}{2}x^TQx+c^Tx+d \le \beta\Sigma_i x_i^2+c^Tx+d, \text{ for all } x\ne 0 \quad (10)$$

with $\alpha=\lambda_{min}/2$, $\beta=\lambda_{max}/2$, obtained by expanding x in the orthonormal eigen basis of Q. These bounds define a convex lower-bound relaxation (coefficient β) and a concave upper-bound relaxation (coefficient α) that are useful in global-optimization schemes.

3.3 Mangasarian–Fromovitz Constraint Qualification

MFCQ holds at a feasible point x^* if the only pair (λ, μ), $\mu\ge 0$, satisfying $h'(x^*)^T\lambda+g'(x^*)^T\mu=0$ and $\mu^Tg(x^*)=0$ is $\lambda=0$, $\mu=0$. MFCQ guarantees the existence of feasible descent directions and rules out degenerate constraint configurations.

Theorem 1 *If a sequence $x^k\to x^*$ generated by the approximate-gradient-projection map $d(x,\gamma)=P_{\Omega(x,\gamma)}(x-\nabla f(x))-x$ satisfies $d(x^k,\gamma)\to 0$, then x^* is a Fritz–John point of problem (7); if MFCQ additionally holds at x^*, then x^* is a KKT point.*

Proof. The projected point y_k solving $\min \tfrac{1}{2}\|y-x_k\|^2+\nabla f(x_k)^T(y-x_k)$ over the linearized set $\Omega(x_k,\gamma)$ yields multipliers (λ_k,μ_k) satisfying the first-order optimality system of the projection. If these multipliers were unbounded, normalizing by $M_k=\|(\lambda_k,\mu_k)\|_\infty$ and passing to a limit point $(\hat\lambda,\hat\mu)$ with $\|(\hat\lambda,\hat\mu)\|_\infty=1$ gives $h'(x^*)^T\hat\lambda+g'(x^*)^T\hat\mu=0$, contradicting MFCQ. Hence the multipliers stay bounded, and any convergent subsequence yields KKT multipliers at x^*.

4. Quadratic Penalty Method

4.1 Formulation

Consider the equality-constrained QOP

$$\text{Min } \Phi(x) = \tfrac{1}{2}x^TQx+b^Tx \text{ s. t. } x\in S=\{x\in R^n : h(x)=Ax-c=0, x\ge 0\} \quad (11)$$

The quadratic penalty method replaces (11) with the unconstrained problem

$$\text{Min } \Phi(x,\rho) = \Phi(x)+(\rho/2)\|Ax-c\|^2 \quad (12)$$

with penalty parameter $\rho\ge 0$ as $\rho\to\infty$ the minimizer x^*_ρ of (12) converges to the solution of (11). For an inequality-constrained problem $Dx\le e$, the corresponding penalty adds $(\rho/2)\max\{0,\|Dx-e\|\}^2$, which penalizes only infeasible points. Comparing the stationary condition of (12) with the KKT stationary condition of (11) gives the multiplier estimate $\lambda^*\approx -\rho\, h(x^*_\rho)$ as $\rho\to\infty$; increasing ρ monotonically decreases constraint violation while monotonically



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increasing the achieved objective value — the classical feasibility/optimality trade-off of penalty methods.

Theorem 2. If A has full row rank and the reduced Hessian Z^TQZ (Z a null-space basis of A) is positive definite, then (11) has a unique global solution x^* satisfying the KKT system, and the associated KKT matrix is non-singular.

Proof. For any feasible x write $x = x^*-p$ with $Ap = 0$, so $p = Zv$ for some v. Expanding $\Phi(x)$ and eliminating the cross term with the KKT stationary condition $Q\,x^*=A^T\lambda^*-b$ gives $\Phi(x)=\tfrac{1}{2}v^TZ^TQZv+\Phi(x^*)$. Since $Z^TQZ\succ 0$, $\Phi(x)>\Phi(x^*)$ for every feasible $x\neq x^*$, proving x^* is the unique global solution.

4.2 Augmented Lagrangian

Exact recovery under the pure penalty method requires $\rho\to\infty$, which causes ill-conditioning. The augmented Lagrangian

$$L(x, \lambda_A, \rho)=\Phi(x)-(Ax-c)^T\lambda_A+(\rho/2)\,\|Ax-c\|^2 \qquad (13)$$

combines the Lagrangian and penalty ideas; minimizing L in x gives the KKT-type linear system

$$[Q\rho\ A^T\,;\,A\ 0]\ [x\,;\,\lambda_A] = [-b+\rho\,A^Tc\,;\,c],\quad Q\rho=Q+\rho\,A^TA \qquad (14)$$

Whose solution satisfies $\|Ax^*\rho-c\|^2 \le (2/\rho)\{\Phi(x^*)-L0(x^*0,\lambda A)\}$, so the feasibility error decreases with ρ without requiring $\rho\to\infty$. The updated multiplier estimate $\tilde{\lambda}A=\lambda A+\rho\,(Ax^*\rho-c)$ is a better approximation of the true Lagrange multiplier, giving the augmented Lagrangian method superior numerical stability compared with the pure penalty method.

5. Quadratic Fractional Optimization Problem (QFOP)

5.1 Formulation

A QFOP optimizes a ratio in which at least one of the numerator or denominator is quadratic, subject to linear constraints — a natural model for efficiency-type objectives such as profit/cost or return/risk.

Two standard forms are

$$\text{Max } \Phi(x) = f(x)/g(x) = (\tfrac{1}{2}x^TQfx+df)\,/\,(\tfrac{1}{2}x^TQgx+c^Tx+dg),\ x\in S \qquad (15)$$

$$\text{Max } \Phi(x) = f(x)/g(x) = (c1^Tx+\alpha)(c2^Tx+\beta)\,/\,[(c3^Tx+\gamma)(c4^Tx+\delta)],\ x\in S \qquad (16)$$

with $S=\{x\in R^n : Ax \le b,\ x\ge 0\}$; equation (15) is Form I (ratio of general quadratics) and equation (16) is Form II (ratio of linear-factorized quadratics).

6. Multi-Objective Quadratic Fractional Optimization Problem (MOQOP)

A Multi-Objective Quadratic Fractional Optimization Problem (MOQFOP) is a nonlinear mathematical programming problem in which several quadratic fractional objective functions are optimized simultaneously over a common feasible set. The mathematical structure combines three important features: quadratic functions, fractional objective functions, and multiple competing optimization criteria. The problem is particularly relevant when a decision-maker must allocate limited resources among several activities while balancing different performance measures, such as cost, efficiency, energy consumption, throughput, and reliability. Unlike a conventional single-objective optimization problem, where the objective is to determine a single optimal solution with respect to one criterion, a multi-objective problem generally seeks a collection of solutions representing different trade-offs among the objectives.

6.1 General Mathematical Formulation

$$\text{Minimize}\quad F(x) = (f_1(x), f_2(x), \ldots, f_p(x))$$

$$f_i(x) = N_i(x) \,/\, D_i(x),\quad i = 1, 2, \ldots, p$$

$$\text{subject to}: \ g_j(x) \le 0,\ j = 1, \ldots, m;\quad h_k(x) = 0,\ k = 1, \ldots, r;\quad x \in X \subseteq \mathbb{R}^n$$

$$N_i(x) = x^T A_i\, x + a_i^T x + \alpha_i$$

$$D_i(x) = x^T B_i\, x + b_i^T x + \beta_i$$



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Here, x = $(x_1, x_2, \ldots x_n)^T$ is the decision vector; p is the number of objectives; n is the number of decision variables; and m and r are the numbers of inequality and equality constraints. A_i and B_i are quadratic coefficient matrices, a_i and b_i are linear coefficient vectors, and α_i and β_i are constants. The matrices may be taken symmetric because only their symmetric parts affect the quadratic forms.

A standard well- defined ness assumption is $D_i(x) > 0$ for every feasible x and each i. This prevents division by zero and supports meaningful ratio comparisons.

6.2 Mathematical Structure of the Feasible Region

$C = \{x \in X : g_j(x) \leq 0, j = 1, \ldots, m; h_k(x) = 0, k = 1, \ldots, r\}$

The feasible region C contains all decision vectors satisfying the constraints. Constraints may be linear or nonlinear. For example, a resource restriction can be written as $\Sigma_l\, c_l\, x_l \leq R$, where c_l is the resource requirement of activity and R is the available resource. Non-negativity restrictions $x_l \geq 0$ may also be imposed. If C is nonempty and compact, and every denominator is strictly positive on C, then each objective is continuous and attains an individual minimum on C. This does not guarantee that all objectives attain their individual minima at the same point.

6.3 Connection with Penalty and Smoothing Methods

Penalty methods can be used when constraints are difficult to enforce directly during numerical optimization. For inequality constraints $g_j(x) \leq 0$, define the violation measure:

$V(x) = \Sigma_{j=1}^{m} [\max\{0, g_j(x)\}]^2$

Then $V(x) \geq 0$, and $V(x) = 0$ exactly when all inequality constraints are satisfied. A penalized scalarization is:

Minimize over $x \in X$: $\Sigma_{i=1}^{p} w_i\, f_i(x) + \rho\, V(x), \quad \rho > 0$

As ρ increases, constraint violations become more costly. Under suitable assumptions and parameter choices, solutions of penalized problems may approach solutions of the constrained problem.

For smoothing, replace max {0,t} by the differentiable approximation:

$\Phi\, \mu(t) = \tfrac{1}{2}[t + \sqrt{(t^2 + \mu^2)}], \quad \mu > 0$

lim as $\mu \to 0^+$ $\varphi\, \mu(t) = \max\{0,t\}$

The smoothed violation function and penalized scalarization are:

$V\, \mu(x) = \Sigma_{j=1}^{m} [\varphi\, \mu(g_j(x))]^2$

Minimize over $x \in X$: $\Sigma_{i=1}^{p} w_i\, f_i(x) + \rho\, V\, \mu(x)$

Here μ controls smoothing accuracy and ρ controls the penalty strength. This is a scalarized, smoothed penalty approach—not automatically an exact solution method for the original vector problem. Convergence to a Pareto-optimal solution requires suitable assumptions on feasibility, objectives, scalarization, and parameter sequences, with approximation errors controlled appropriately.

For equality constraints, one may add squared residuals:

$V\, \mu(x) = \Sigma_{j=1}^{m} [\varphi\, \mu(g_j(x))]^2 + \Sigma_{k=1}^{r} [h_k(x)]^2$

The exact penalty construction should be selected according to the constraint structure and the desired convergence properties.

7. COMPARISON AND DISCUSSION

The three methodologies developed above differ fundamentally in structure. The quadratic penalty method (Section 4) is a solution technique for single-objective constrained problems, converting constraints into penalty terms; its strength is theoretical clarity and broad applicability, at the cost of ill-conditioning for large ρ. QFOP (Section 5) is a problem class in its own right — single-objective but nonlinear and fractional — well suited to efficiency-type objectives in economics, finance and engineering, though generally non-convex. MOQOP (Section 6) is the most



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general of the three, addressing multiple simultaneous ratio objectives through Pareto fronts rather than single optimal points, which is computationally the most expensive but yields the richest decision support. Table 1 summarizes the comparison.

Table 1: Comparison of the Quadratic Penalty Method, QFOP, and MOQOP

Aspect	Quadratic Penalty Method	QFOP	MOQOP
Objective structure	Single quadratic objective, equality/inequality constraints	Single ratio of quadratic functions	Multiple simultaneous ratio objectives
Solution approach	Sequence of unconstrained minimizations, $\rho \to \infty$; augmented Lagrangian refinement	Fuzzy goal programming on numerator/negated-denominator pair	FGP extended to p objective pairs; Pareto optimality
Computational effort	One unconstrained solve per ρ	Sequence of QP sub problems	Multiple runs; Pareto-front generation
Main drawback	Ill-conditioning for large ρ	Generally non-convex; sensitive to feasible region	High cost; subjective preference needed to pick a Pareto point
Typical applications	Structural design, control, constrained QP	Economics, finance, resource allocation	Portfolio optimization, planning, engineering design

8. CONCLUSION

The quadratic penalty method, the quadratic fractional optimization problem, and the multi-objective quadratic optimization problem represent three complementary strands of quadratic-optimization theory. The penalty method converts constrained problems into an unconstrained sequence at the cost of ill-conditioning for large penalty parameters; QFOP captures efficiency-type ratio objectives useful in economics, finance and engineering but is inherently non-convex and analytically demanding; and MOQOP extends this idea to multiple simultaneous ratio objectives, offering rich Pareto trade-off information at significant computational cost. Because these techniques are complementary rather than competing — penalty ideas are routinely embedded within fractional and multi-objective solution procedures — together they provide a coherent mathematical toolkit for constrained decision-making problems arising in engineering, economics, and data-driven applications.



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